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ON CERTAIN SEQUENCES OF INTEGERS

BY

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ABSTRACT. Let the sequence $\{k_i\}$ satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots$. Then, under certain conditions satisfied by $\{k_i\}$, it is shown that there exists an integer s such that the sequence of integers of the form $x_1^{k_1} + \cdots + x_s^{k_s}$ has positive density. Also, some special sequences having positive densities are constructed.

1. **Introduction.** In an earlier paper (see [2]), I considered the sequences of integers of the form

$$x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}, \tag{1}$$

where the x's are nonnegative integers, and k's are natural numbers satisfying $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$. Let $U_s(k_1, \cdots, k_s; N)$ denote the number of integers of the form (1) that are less than N, N being a given large natural number. The exact order of magnitude of $U_s(k_1, \cdots, k_s; N)$, namely $N^{1/k_1 + 1/k_2 + \cdots + 1/k_s}$, was determined when the k's satisfy the conditions

$$2 \leqslant k_1 < k_2 < \cdots < k_s,$$

and

$$1/k_i > 1/k_{i+1} + \cdots + 1/k_s \qquad (i = 1, 2, \cdots, s-1). \tag{2}$$

However, condition (2) makes the sequence $\{k_i\}$ somewhat thin, in the sense that $\Sigma_{i=1}^{s} (1/k_i) < 1$. Consequently, asymptotic density of the sequence (1) is zero.

In this paper, we construct some sequences of the form (1) which have positive densities. We state a general theorem and also some special cases which are illustrative.

THEOREM 1. *Let the sequence $\{k_i\}$ satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots$, and be such that there exist integers l, m such that*

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(3) $$\mu(l) = \sum_{i=1}^{l} \frac{1}{k_i} > 1,$$

and

(4) $$\prod_{i=l+1}^{l+m} \left(1 - \frac{1}{k_i}\right) < 2\sigma(l),$$

where $\sigma(l) = \Sigma_{i=1}^{l} \rho_i'$, *with*

(5) $$\rho_i' = \begin{cases} \dfrac{1}{k_i} \min\left(\dfrac{1}{2^{k_i-1}} - \delta, \dfrac{1}{k_l}\left(1 - \dfrac{1}{k_l}\right)\right), & 2 \leqslant k_i \leqslant 11, \\ \dfrac{1}{k_l}\left(1 - \dfrac{1}{k_l}\right)(9k_i^2 \log k_i)^{-1}, & k_i \geqslant 12, \end{cases}$$

δ *being a small positive constant. Write* $s = l + m$. *(The existence of* l *and* m *will be ensured if* $\Sigma_{i=1}^{\infty} (1/k_i) = \infty$.) *Then, the sequence* $x_1^{k_1} + \cdots + x_s^{k_s}$ *has positive Schnirelmann density.*

THEOREM 2. *The following sequences have positive Schnirelmann densities*

(6) $$x_1^2 + x_2^3 + x_3^5 + x_4^k \qquad (k \geqslant 5),$$

(7) $$x_1^2 + x_2^3 + x_3^6 + x_4^k \qquad (k \geqslant 6),$$

(8) $$x_1^2 + x_2^4 + x_3^8 + \cdots + x_s^{2^s} + x_{s+1}^{2^s} + x_{s+2}^k,$$

(for any s *and* $k \geqslant 2^s$).

REMARK 1. The bounds for k in (6), (7) and (8) are not required, but for smaller k the results, while still true, are less deep. Hence, (7) and (8) represent in a sense best possible results of their kind since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1, \quad \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^s} + \frac{1}{2^s} = 1,$$

and k is arbitrary. Other similar combinations of the exponents can also be found.

REMARK 4. In order to prove the theorems, it is clearly sufficient to show that the lower asymptotic densities are positive.

2. **Proof of Theorem 1.** We choose l, m in accordance with (3) and (4), and define

$$X_j = \{cN^{(1-1/k_{j+1})\cdots(1-1/k_{l+m})}\}^{1/k_j} \qquad (j = l+1, \cdots, l+m)$$

with $c = (m2^{k_{l+m}})^{-k_{l+m}}$;

$$\Lambda(\alpha) = \prod_{i=1}^{l} \left\{ \sum_{1 < x^{k_i} \leqslant N/4l} \exp(2\pi i \alpha x^{k_i}) \right\};$$

$$\Omega(\alpha) = \prod_{j=l+1}^{l+m} \left\{ \sum_{X_j < x < 2X_j} \exp(2\pi i \alpha x^{k_i}) \right\}.$$

Then, by Lemma 2 of (3),

$$(9) \qquad N^{1-\Pi_{j=l+1}^{l+m}(1-1/k_j)} \ll X_{l+1}X_{l+2}\cdots X_{l+m} \ll N^{1-\Pi_{j=l+1}^{l+m}(1-1/k_j)}$$

and, if $X_j \leqslant x_j \leqslant 2X_j \quad (j = l+1, \cdots, l+m)$,

$$(10) \qquad cN < x_{l+1}^{k_{l+1}} + \cdots + x_{l+m}^{k_{l+m}} < N/3.$$

Also, taking note of the definitions of $\Lambda(\alpha)$ and $\Omega(\alpha)$, and allowing for change in the implied constant, we have by Theorem A of (3), that

$$(11) \qquad \int_0^1 |\Lambda(\alpha)\Omega(\alpha)|^2 d\alpha \ll N^{-1}\{\Lambda(0)\Omega(0)\}^2.$$

Let $r(m)$ denote the number of representations of m in the form

$$(12) \qquad m = x_1^{k_1} + \cdots + x_l^{k_l} + x_{l+1}^{k_{l+1}} + \cdots + x_{l+m}^{k_{l+m}}$$

subject to

$$(13) \qquad 1 \leqslant x_i \leqslant (N/4)^{1/k_i} \quad (i = 1, \cdots, l) \quad \text{and} \quad X_j \leqslant x_j \leqslant 2X_j$$

$$\sum_m r^2(m) = \int_0^1 |\Lambda(\alpha)\Omega(\alpha)|^2 d\alpha, \qquad (j = l+1, \cdots, l+m)$$

and thus, by (11),

$$(15) \qquad \sum_m r^2(m) \ll N^{-1}\{\Lambda(0)\Omega(0)\}^2.$$

We also have

$$\sum_m r(m) \gg \{N^{1/k_1} \cdots N^{1/k_l}\}\{X_{l+1} \cdots X_{l+m}\} \tag{16}$$
$$\gg \Lambda(0)\Omega(0)$$

since $N^{1/k_1} \cdots N^{1/k_l} \gg \Lambda(0)$, and $X_{l+1} \cdots X_{l+m} \gg \Omega(0)$. Hence, with $s = l + m$,

$$U_s(k_1, \cdots, k_s; N) \gg \sum_{m;r(m)>0} 1$$
$$\gg \left\{\sum_m r(m)\right\}^2 \Big/ \left\{\sum_m r^2(m)\right\} \gg N$$

by Cauchy's inequality.

Thus, there exists a constant $c > 0$ depending at most on $k_1, \cdots, k_s$ such that for sufficiently large N,

$$U_s(k_1, \cdots, k_s; N) > cN. \tag{17}$$

Theorem 1 now follows since 1 belongs to the sequence $x_1^{k_1} + \cdots + x_s^{k_s}$.

3. **Proof of Theorem 2.** First, we prove two auxiliary results.

LEMMA 1. *Let* ϵ *be a sufficiently small fixed positive number. Then*, (i) *the number of solutions of*

$$x_1^2 + x_2^3 + x_3^5 = y_1^2 + y_2^3 + y_3^5 \tag{18}$$

with $x_1, y_1 < (N/4)^{1/2}; x_2, y_2 < (N/4)^{1/3}; x_3, y_3 < (N/4)^{1/5}$ *is*

$$\ll N^{-1+\epsilon} N^{2(1/2+1/3+1/5)};$$

(ii) *the number of solutions of*

$$x_1^2 + x_2^3 + x_3^6 = y_1^2 + y_2^3 + y_3^6 \tag{19}$$

with $x_1, y_1 < (N/4)^{1/2}; x_2, y_2 < (N/4)^{1/3}; x_3, y_3 < (N/4)^{1/6}$ *is*

$$\ll N^{-1+\epsilon} N^{2(1/2+1/3+1/6)};$$

(iii) *the number of solutions of*

$$x_1^2 + x_2^4 + \cdots + x_s^{2^s} + x_{s+1}^{2^s} = y_1^2 + y_2^4 + \cdots + y_s^{2^s} + y_{s+1}^{2^s} \tag{20}$$

with $x_i, y_i < (N/(s+2))^{1/2^i}$ $(i = 1, \cdots, s)$; $x_{s+1}, y_{s+1} < (N/(s+2))^{1/2^s}$ *is* $\ll N^{-1+\epsilon} N^{2(1/2+\cdots+1/2^s+1/2^s)}$.

PROOF. We prove only (i), since (ii) and (iii) are proved in the same way.

The number of solutions of $x_1^2 - y_1^2 = n$ with $x_1 \neq y_1$ is $\ll |n|^\epsilon$. Hence, the number of solutions of (18) (by writing it in the form $x_1^2 - y_1^2 = y_2^3 + y_3^5 - x_2^3 - x_3^5$) with $x_1 \neq y_1$ is

$$\ll N^{2(1/3+1/5)+\epsilon}. \tag{21}$$

By the same argument, we see that the number of solutions (18) with $x_1 = y_1, x_2 \neq y_2$ is

$$\ll N^{1/2} \cdot N^{2(1/5)+\epsilon}; \tag{22}$$

the number of solutions of (18) with $x_1 = y_1, x_2 = y_2, x_3 = y_3$ is

$$\ll N^{1/2+1/3+1/5}. \tag{23}$$

From (21), (22) and (23) we get (i).

(ii) follows in the same way on noting that

$$2(1/2 + 1/3 + 1/6) \geqslant \text{any of } 2(1/3 + 1/6) + 1,\ \ 1/2 + 2(1/6) + 1.$$

Similar inequalities establish (iii).

LEMMA 2. *Let*

$$\begin{aligned} F_1(\alpha) = &\left\{\sum_{1<x^2<N/4} \exp(2\pi i\alpha x^2)\right\}\left\{\sum_{1<x^3<N/4} \exp(2\pi i\alpha x^3)\right\} \\ &\times\left\{\sum_{1<x^5<N/4} \exp(2\pi i\alpha x^5)\right\}\left\{\sum_{1<x^k<N/4} \exp(2\pi i\alpha x^k)\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} F_2(\alpha) = &\left\{\sum_{1<x^2<N/4} \exp(2\pi i\alpha x^2)\right\}\left\{\sum_{1<x^3<N/4} \exp(2\pi i\alpha x^3)\right\} \\ &\times\left\{\sum_{1<x^6<N/4} \exp(2\pi i\alpha x^6)\right\}\left\{\sum_{1<x^k<N/4} \exp(2\pi i\alpha x^k)\right\}, \end{aligned} \tag{25}$$

$$\begin{aligned} F_3(\alpha) = &\left(\prod_{j=1}^{s}\left\{\sum_{1<x^{2^j}<N/(s+2)} \exp(2\pi i\alpha x^{2^j})\right\}\right) \\ &\times\left\{\sum_{1<x^{2^s}<N/(s+2)} \exp(2\pi i\alpha x^{2^s})\right\}\left\{\sum_{1<x^k<N/(s+2)} \exp(2\pi i\alpha x^k)\right\}. \end{aligned} \tag{26}$$

Then,

$$\int_0^1 |F_1(\alpha)|^2 \, d\alpha \ll N^{-1}\{F_1(0)\}^2, \tag{27}$$

$$\int_0^1 |F_2(\alpha)|^2 \, d\alpha \ll N^{-1}\{F_2(0)\}^2, \tag{28}$$

and

$$\int_0^1 |F_3(\alpha)|^2 \, d\alpha \ll N^{-1}\{F_3(0)\}^2.$$

PROOF. In view of Lemma 1, the proofs of (27), (28) and (29) are similar; and so we prove only (27).

As in (3), we subdivide the interval $Q^{-1} \leqslant \alpha \leqslant 1 + Q^{-1}$ with $Q = 2k\,[N^{1/k}]^{k-1}$ into basic intervals D and supplementary intervals E; so that $D \cup E = [Q^{-1}, 1 + Q^{-1}]$. Since $1/2 + 1/3 + 1/5 + 1/k > 1$, it follows as in Lemma 9 of (3), that

$$\int_D |F_1(\alpha)|^2 \, d\alpha \ll N^{-1}(F_1(0))^2.$$

As for the integral over E, we proceed thus:

Let ρ' be defined by (5) with $k_i = k_l = k$. Then,

$$f(\alpha) = \left(\sum_{1 < x^k \leqslant N/4} \exp(2\pi i \alpha x^k) \right) \ll N^{1/k - \rho'}. \tag{30}$$

Also, if

$$F(\alpha) = \left(\sum_{1 < x^2 \leqslant N/4} \exp(2\pi i\alpha x^2) \right) \left(\sum_{1 < x^3 \leqslant N/4} \exp(2\pi i\alpha x^3) \right) \left(\sum_{1 < x^5 \leqslant N/4} \exp(2\pi i\alpha x^5) \right),$$

then $\int_{Q^{-1}}^{1+Q^{-1}} |F(\alpha)|^2 \, d\alpha = \int_0^1 |F(\alpha)|^2 \, d\alpha$ is the number of solutions of (18), and hence $\ll N^{-1+\epsilon} \cdot N^{2(1/2+1/3+1/5)}$. Thus, by (30),

$$\int_E |F_1(\alpha)|^2 \, d\alpha \leqslant \left\{ \max_{\alpha \in E} |f(\alpha)|^2 \right\} \int_{Q^{-1}}^{1+Q^{-1}} |F(\alpha)|^2 d\alpha$$

$$\ll N^{2/k - 2\rho'} \cdot N^{-1+\epsilon} \cdot N^{2(1/2+1/3+1/5)}$$

$$\ll N^{-1} N^{2(1/2+1/3+1/5+1/k)} \ll N^{-1}(F_1(0))^2.$$

This completes the proof of (27).

Theorem 2 can be deduced from (27), (28) and 29 in the same way as Theorem 1 was deduced from (11).

REMARK 5. The method of proof of Theorem 2 can be used for various other combinations of positive integral powers. Thus, for example, one can show that for any $k \geqslant 2$, the following sequences have positive Schnirelmann densities.

(a) $x_1^2 + x_2^3 + x_3^7 + x_4^{42} + x_5^k$

(b) $x_1^2 + x_2^3 + x_3^8 + x_4^{24} + x_5^k$

(c) $x_1^2 + x_2^3 + x_3^9 + x_4^{18} + x_5^k$

(d) $x_1^2 + x_2^3 + x_3^{10} + x_4^{15} + x_5^k$

(e) $x_1^2 + x_2^3 + x_3^{11} + x_4^{13} + x_5^k$

(f) $x_1^2 + x_2^3 + x_3^{12} + x_4^{12} + x_5^k$

(g) $x_1^2 + x_2^4 + x_3^5 + x_4^{20} + x_5^k$

(h) $x_1^2 + x_2^4 + x_3^6 + x_4^{12} + x_5^k$

(i) $x_1^2 + x_2^4 + x_3^7 + x_4^9 + x_5^k$

(j) $x_1^2 + x_2^4 + x_3^8 + x_4^9 + x_5^{72} + x_6^k$

(k) $x_1^2 + x_2^4 + x_3^8 + x_4^{10} + x_5^{40} + x_6^k$

(l) $x_1^2 + x_2^4 + x_3^8 + x_4^{11} + x_5^{29} + x_6^k$

(m) $x_1^2 + x_2^4 + x_3^8 + x_4^{12} + x_5^{24} + x_6^k$

(n) $x_1^2 + x_2^4 + x_3^8 + x_4^{13} + x_5^{20} + x_6^k$

(0) $x_1^2 + x_2^4 + x_3^8 + x_4^{14} + x_5^{18} + x_6^k$

(p) $x_1^2 + x_2^4 + x_3^8 + x_4^{15} + x_5^{17} + x_6^k$

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